B.Sc. (H) Mathematics Semester -II

Paper : Real Analysis [16/03/2020 - 20/03/2020]

Book : Bartle & Sherbert, Donald R., "Introduction to Real Analysis."

Section 3.5 The Cauchy Criterion

Definition:- A sequence $X = \langle x_n \rangle$ of real numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \geq H$,

$$|x_n - x_m| < \varepsilon.$$

Example 3.5.2 (a). Show that the sequence $\langle \frac{1}{n} \rangle$ is a Cauchy sequence.

Soln:- If $\varepsilon > 0$ be given, choose a natural number H such that $H > \frac{2}{\varepsilon}$.

If $m, n \geq H$, we have $\frac{1}{n} \leq \frac{1}{H} < \frac{\varepsilon}{2}$

and $\frac{1}{m} \leq \frac{1}{H} < \frac{\varepsilon}{2}$.

Therefore, if $n, m \geq H$, then

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right|$$

$$\leq \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$|x_n - x_m| < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\langle \frac{1}{n} \rangle$ is a Cauchy sequence.

Example 3.5.2 (b) Show that $\langle 1 + (-1)^n \rangle$ is not a Cauchy sequence.

Solution:- Let $\langle x_n \rangle = \langle 1 + (-1)^n \rangle$

If n is even, then $x_n = 2$ and $x_{n+1} = 0$.

~~If n is odd, then x~~

Take $\varepsilon_0 = 2$, then for any H we can choose an even number $n > H$ and let $m = n+1$

then $|x_n - x_m| = |x_n - x_{n+1}|$

$$= |2 - 0| = 2 = \varepsilon_0$$

so $|x_n - x_m| \nless \varepsilon_0$

Therefore, $\langle x_n \rangle$ is not a Cauchy sequence. //

<u>Remark</u>: The negation of the definition of Cauchy sequence is: There exists $\varepsilon_0 > 0$ such that for every H there exist at least one $n > H$ and at least one $m > H$ such that $|x_n - x_m| \geq \varepsilon_0$.

<u>Lemma 3.5.4</u> A Cauchy sequence of real numbers is bounded.

<u>Proof</u>: Let $X = \langle x_n \rangle$ be a Cauchy sequence and let $\varepsilon = 1$. If $H = H(1)$ and $n \geq H$, then by the def^n of Cauchy sequence

$$|x_n - x_H| < 1.$$

By the triangle, we have $|x_n| \leq |x_H| + 1 \;\; \forall\, n \geq H$

If we set $M = \sup\{|x_1|, |x_2|, \ldots, |x_{H-1}|, |x_H| + 1\}$

then $|x_n| \leq M \;\; \forall\, n \in \mathbb{N}$.

Hence, $\langle x_n \rangle$ is a bounded sequence.

<u>Cauchy Convergence Criterion</u> A sequence of real numbers is convergent if and only if it is a Cauchy sequence

Proof:- Let $X = \langle x_n \rangle$ is a convergent sequence and let $x = \lim X$, then given $\varepsilon > 0$ there is a natural number $K(\frac{\varepsilon}{2})$ such that if $n \geq K(\frac{\varepsilon}{2})$ then $|x_n - x| < \frac{\varepsilon}{2}$.

Thus, if $H(\varepsilon) = K(\frac{\varepsilon}{2})$ and if $n, m \geq H(\varepsilon)$, then we have

$$|x_n - x_m| = |(x_n - x) + (x - x_m)|$$

$$\leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$|x_n - x_m| < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\langle x_n \rangle$ is a Cauchy sequence.

Conversely, let $X = \langle x_n \rangle$ be a Cauchy sequence, then by lemma (3.5.4) X is bounded.

By Bolzano - Weierstrass theorem, there is a subsequence $X' = \langle x_{n_k} \rangle$ of X that converges to some real number x^*.

We will show that X converges to x^*.

Since $X = \langle x_n \rangle$ is a Cauchy sequence, given $\varepsilon > 0$ there is a natural number $H(\frac{\varepsilon}{2})$ such that if $n, m \geq H(\frac{\varepsilon}{2})$ then

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \text{——— (1)}$$

Since the subsequence $X' = \langle x_{n_k} \rangle$ converges to x^*, there is a natural number $K \geq H(\frac{\varepsilon}{2})$ belonging to the set $\{n_1, n_2, \ldots\}$ such that

$$|x_K - x^*| < \frac{\varepsilon}{2}.$$

Since $K \geq H(\frac{\varepsilon}{2})$, from (1) with $m = K$

$$|x_n - x_K| < \frac{\varepsilon}{2} \quad \text{for } n \geq H(\tfrac{\varepsilon}{2})$$

Therefore, if $n \geq H(\frac{\varepsilon}{2})$, we have

$$|x_n - x^*| = |(x_n - x_k) + (x_k - x^*)|$$
$$\leq |x_n - x_k| + |x_k - x^*|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$|x_n - x^*| < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim(x_n) = x^*$.
Therefore the sequence X is convergent.

Example – 3.5.6 Show that the sequence $X = \langle x_n \rangle$ defined by

$x_1 = 1$, $x_2 = 2$, and $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$ for $n > 2$

is convergent. Find its limit.

Solⁿ:- By principal of mathematical Induction, one can easily establish that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$. (??)

Also $|x_n - x_{n+1}| = \frac{1}{2^{n-1}}$ for $n \in \mathbb{N}$

Thus, if $m > n$, we may employ the triangle Inequality to obtain

$$|x_n - x_m| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$$
$$= \frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{m-2}}$$
$$= \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n-1}}\right) < \frac{1}{2^{n-2}}$$

Therefore, given $\varepsilon > 0$, if n is chosen so large that $\frac{1}{2^n} < \varepsilon$ and if $m \geq n$, then it follows that

$$|x_n - x_m| < \varepsilon.$$

Therefore X is a Cauchy sequence in $\mathbb{R}$.
By Cauchy convergence criterion, the sequence X is convergent.

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Example:- Let $X = \langle x_n \rangle$ be a sequence of real numbers given by

$$x_1 = \frac{1}{1!},\ x_2 = \frac{1}{1!} - \frac{1}{2!} + \cdots, \ \cdots$$

$$x_n = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!}, \cdots$$

Show that the sequence $\langle x_n \rangle$ is convergent.

Soln Do it by yourself.

Example. Show that the sequence $\langle x_n \rangle = \langle 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \rangle$ is divergent.

Soln:-

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

let $m > n$, $x_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m}$

$$x_m - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$$

Since $n+1 < m$, $n+2 < m$, $\cdots$

so $\frac{1}{n+1} > \frac{1}{m}$, $\frac{1}{n+2} > \frac{1}{m}$, $\cdots$

$$x_m - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m} > \underbrace{\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}}_{(m-n) \text{ terms}}$$

$$x_m - x_n > \frac{(m-n)}{m} = 1 - \frac{n}{m}$$

In particular, if $m = 2n$ we have

$$x_{2n} - x_n > \frac{1}{2}.$$

This shows that $\langle x_n \rangle$ is not a Cauchy sequence.

Exercise - 3.5

Q.1 Give an example of a bounded sequence that is not a Cauchy sequence.

Sol^n $\langle(-1)^n\rangle$

Q.2 Show directly from the definition that the following are Cauchy sequences.

(a) $\langle \frac{n+1}{n} \rangle$ (b) $\langle 1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \rangle$

Soln:→ (a) let $\varepsilon>0$ be given.

Let $\langle x_n\rangle = \langle \frac{n+1}{n} \rangle = \langle 1+\frac{1}{n} \rangle$

Choose a natural number H such that $H > \frac{2}{\varepsilon}$

Then if $m,n \geq H$ and let $m>n$, we have $\frac{1}{n} \leq \frac{1}{H} < \frac{\varepsilon}{2}$

$$|x_m - x_n| = \left| \left(1+\frac{1}{m}\right) - \left(1+\frac{1}{n}\right) \right|$$

$$= \left| \frac{1}{m} - \frac{1}{n} \right| \leq \frac{1}{m} + \frac{1}{n} = \frac{2}{n}$$

$$|x_m - x_n| < \frac{2}{n} < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

$$|x_m - x_n| < \varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we conclude that $\langle \frac{n+1}{n} \rangle$ is a Cauchy sequence.

(b) Do it by yourself as part (a)

Q-3 Show directly from the definition that the following are not Cauchy sequence.

(a) $\langle (-1)^n \rangle$ (b) $\langle n + \frac{(-1)^n}{n} \rangle$ (c) $\langle \ln n \rangle$

Sol^n (a) Let $x_n = (-1)^n$

If n is even, then $x_n = 1$ and $x_{n+1} = -1$

Take $\varepsilon_0 = 2$, then for any H we can choose an even number $n > H$ and let $m = n+1$

then $|x_n - x_m| = |x_n - x_{n+1}|$

$= |1 - (-1)| = 2 = \varepsilon_0$

So $|x_m - x_n| \nless \varepsilon_0$

Therefore, $\langle x_n \rangle$ is not a Cauchy sequence.

(b), (c) Similar as part (a)

Q-4 If $\langle x_n \rangle$ and $\langle y_n \rangle$ are Cauchy sequences, then $\langle x_n + y_n \rangle$ and $\langle x_n y_n \rangle$ are also Cauchy sequences.

Sol^n:- Apply def^n of Cauchy sequence.

Q-5 If $x_n = \sqrt{n}$, show that $\langle x_n \rangle$ satisfies $\lim |x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.

Sol^n:- Take $m = 4n$ in the def^n of Cauchy sequence and $\varepsilon_0 = 2$.

Q-6 Let p be a given natural number. Give an example of a sequence $\langle x_n \rangle$ that is not a Cauchy sequence, but that satisfies $\lim |x_{n+p} - x_n| = 0$

Sol^n Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which is not a Cauchy sequence (prove it).

However, for any $p \in \mathbb{N}$

$$x_{n+p} - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} \leq \underbrace{\frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1}}_{p\text{-times}}$$

$$x_{n+p} - x_n = \frac{p}{n+1}$$

$$\lim |x_{n+p} - x_n| \leq \lim \frac{p}{n+1}$$

$$\boxed{\lim |x_{n+p} - x_n| = 0}$$

Q-9 If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that $\langle x_n \rangle$ is a Cauchy sequence.

Sol: Since $|x_{n+1} - x_n| < r^n$

If $m > n$, then $|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n|$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$< r^{m-1} + r^{m-2} + \cdots + r^{n+1} + r^n$$

$|x_m - x_n| \leq \dfrac{r^n}{1-r}$, which converges to 0 since $0 < r < 1$.

Q-10, 11 do it as done in example.